Chapter 16 - **NEUTRON FOCUSING LENSES**

Neutron lenses are used to focus neutron beams. They increase intensity on the sample and shrink the neutron spot size on the detector therefore reducing the minimum Q. The effects of focusing lenses on SANS resolution are discussed.

1. FOCUSSING LENSES' BASIC EQUATIONS

The focusing lenses' basic equations are described here (Mildner et al, 2005; Hammouda-Mildner, 2007). The focal length for a set of N lenses of radius of curvature R and index of refraction n is given by:

$$f = \frac{R}{2N(1-n)}. \qquad (1)$$

The index of refraction n is related to the material atomic density ρ, neutron scattering length b, and neutron wavelength λ as:

$$n = 1 - \frac{\rho b}{2\pi}\lambda^2. \qquad (2)$$

The focal length f is also related to the source-to-lenses distance L_1 and lenses-to-image distance L_4 as:

$$\frac{1}{f} = \frac{1}{L_1} + \frac{1}{L_4}. \qquad (3)$$

Combining the above two equations gives a relationship between the number N of lenses used and the neutron wavelength λ for an optimized instrument configuration where the detector is located at the focal spot.

$$\frac{\pi}{\rho b_c}\frac{R}{N\lambda^2} = \frac{L_1 L_4}{L_1 + L_4}. \qquad (4)$$

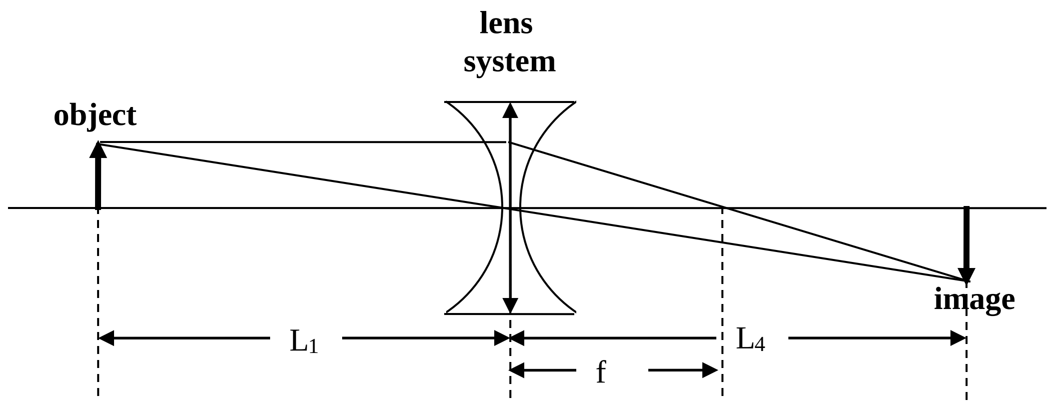

Figure 1: Schematic representation of a focusing lens system showing an object (the neutron source aperture) and its image (on the detector plane). L_1 and L_4 are the source-to-sample and sample-to-detector distances and f is the focal length. In practice, neutron focusing devices comprise many lenses used together.

For MgF_2 lenses, one has:

$$\rho b/\pi = 1.632 * 10^{-6}\,\text{Å}^{-2}$$

so that:

$$\frac{N\lambda^2}{R}\left(\frac{L_1 L_4}{L_1 + L_4}\right) = \frac{\pi}{\rho b} = 6.13 * 10^5\,\text{Å}^2\,. \qquad (5)$$

Consider lenses of radius of curvature R = 2.5 cm and height H = 2.5 cm that are thin at the center (1 mm thickness) in order to keep neutron transmission high. Source-to-sample and sample-to-image distances corresponding to the following SANS instrument configuration (L_1 = 16.14 m, L_4 = 13.19 m) give a focal length of

$$f = \left(\frac{L_1 L_4}{L_1 + L_4}\right) = 726\,\text{cm}\,. \qquad (6)$$

This gives $N\lambda^2$ = 2111 Å^2. The use of 7 consecutive lenses (N = 7) focuses neutrons of wavelength λ = 17.36 Å with a focal distance of 726 cm. The use of 30 consecutive lenses focuses neutrons of wavelength λ = 8.39 Å down to the same focal spot. The use of 14 consecutive lenses corresponds to a focusing wavelength λ = 12.20 Å.

For MgF_2, the index of refraction is:

$$n = 1 - 0.816 * 10^{-6}\lambda^2\,. \qquad (7)$$

Note that the index of refraction of MgF_2 for neutrons is less than unity so that concave lenses focus neutrons whereas convex lenses defocus them. This is opposite to basic optics for light whereby the index of refraction is greater than unity.

2. RESOLUTION WITH FOCUSSING LENSES

Consider a neutron beam with a triangular wavelength distribution and a focusing lens system optimized for the main wavelength λ_0 in that distribution. The main focal length is noted f_0 and corresponds to object-to-lens and lens-to-image distances of L_1 and L_2 respectively. Moreover, consider another wavelength λ within the same distribution and its corresponding focal length f. The object-to-lens and lens-to-image distances are L_1 and L_4 respectively for this wavelength.

$$\frac{1}{f_0} = \frac{1}{L_1} + \frac{1}{L_2} = \frac{2N}{R}\frac{\rho b}{2\pi}\lambda_0{}^2 . \quad (8)$$

$$\frac{1}{f} = \frac{1}{L_1} + \frac{1}{L_4} = \frac{2N}{R}\frac{\rho b}{2\pi}\lambda^2 .$$

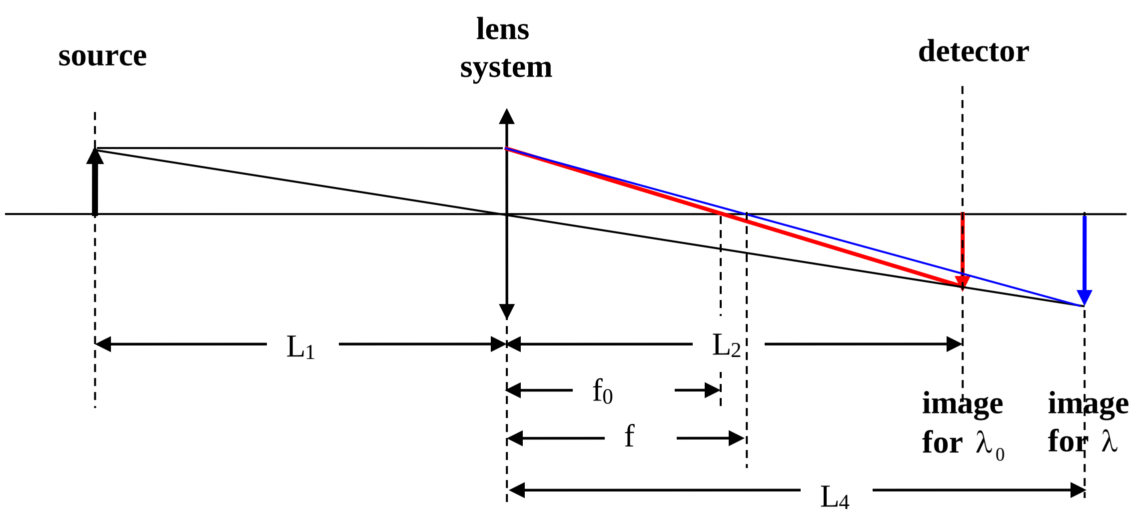

Figure 2: SANS focusing system showing the main image of the neutron source corresponding to the main neutron wavelength λ_0 and another image corresponding to another wavelength λ.

In order to calculate the resolution with the lens system, the "geometry" contribution contains three terms: one that corresponds to the image of the source aperture; another that corresponds to the sample aperture and a term that corresponds to averaging over a detector cell.

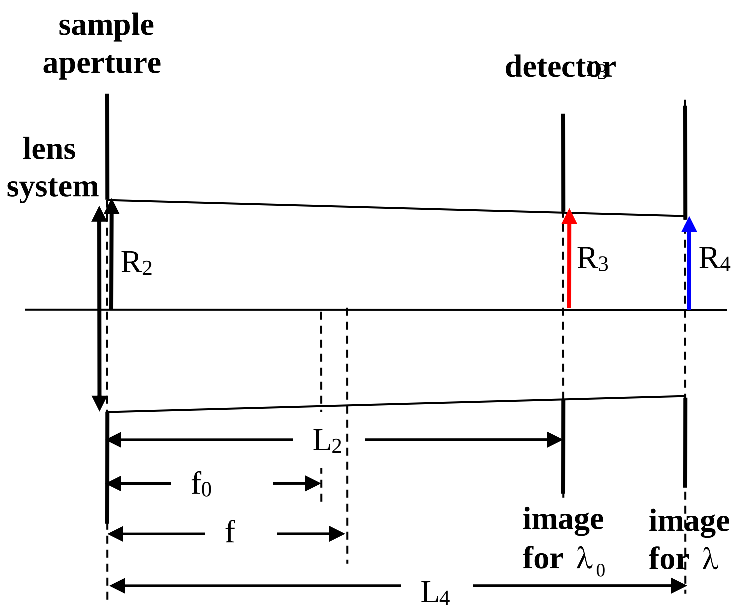

Figure 3: Schematic representation of the three main vertical planes containing the sample aperture, the area detector (source aperture image for λ_0) and the source aperture image for another wavelength λ.

Projection of the "geometry" part of the spatial resolution onto the detector plane in the horizontal direction is expressed as σ_x^2:

$$\left[\sigma_x^2\right]_{geo} = \left(\frac{L_2}{L_4}\right)^2 \frac{R_4^2}{4} + \left(\frac{L_4 - L_2}{L_4}\right)^2 \frac{R_2^2}{4} + \frac{1}{3}\left(\frac{\Delta x_3}{2}\right)^2. \qquad (9)$$

Here R_4 is the radius of the image of the source aperture for the focal length f at wavelength λ. In order to see how the two scale factors were derived, consider the case $R_2 = 0$ for which $R_3 = \frac{L_2}{L_4} R_4$, then the case of $R_4 = 0$ for which $R_3 = \left(\frac{L_4 - L_2}{L_4}\right) R_2$. Δx_3 is the detector cell horizontal size. The image of the source aperture is given by $R_4 = \frac{L_4}{L_1} R_1$.

From the focusing equations, one obtains:

$$\frac{1}{f_0}-\frac{1}{f}=\frac{1}{L_2}-\frac{1}{L_4}=\frac{L_4-L_2}{L_2 L_4}=\frac{1}{f_0}\left(1-\left(\frac{\lambda}{\lambda_0}\right)^2\right). \qquad (10)$$

Therefore:

$$\left[\sigma_x{}^2\right]_{geo}=\left(\frac{L_2}{L_1}\right)^2\frac{R_1{}^2}{4}+\left(\frac{L_1+L_2}{L_1}\right)^2\left(1-\left(\frac{\lambda}{\lambda_0}\right)^2\right)^2\frac{R_2{}^2}{4}+\frac{\Delta x_3{}^2}{12}. \qquad (11)$$

This is the result valid for any wavelength λ. Around the focal wavelength λ_0, the averaging over the triangular wavelength distribution yields for the square term $\left(1-\left(\frac{\lambda}{\lambda_0}\right)^2\right)^2$ the result of $\frac{2}{3}\left(\frac{\Delta\lambda}{\lambda}\right)^2$. Even though the subscript on λ_0 is dropped, it should be remembered that these results are valid only for the focusing wavelength.

Finally:

$$\left[\sigma_y{}^2\right]_{geo}=\left(\frac{L_2}{L_1}\right)^2\frac{R_1{}^2}{4}+\left(\frac{L_1+L_2}{L_1}\right)^2\frac{2}{3}\left(\frac{\Delta\lambda}{\lambda}\right)^2\frac{R_2{}^2}{4}+\frac{\Delta x_3{}^2}{12}. \qquad (12)$$

The spatial resolution in the vertical direction $\sigma_y{}^2$ involves the same terms as $\sigma_x{}^2$ along with contributions due to the gravity effect.

Neutrons follow a parabolic trajectory, which at the detector position (for $z = L_1+L_2$) is given by:

$$y(L_1+L_2)=y_0-A\lambda^2 \qquad \text{where } A=L_2(L_1+L_2)\frac{gm^2}{2h^2}. \qquad (13)$$

The effect of gravity and wavelength spread contribute terms of the following form to σ_y^2:

$$\left[\sigma_{Qy}{}^2\right]_{grav}=<y(z)^2>-<y(z)>^2=A^2[<\lambda^2>-\lambda^2]=A^2\lambda^4\frac{2}{3}\left(\frac{\Delta\lambda}{\lambda}\right)^2. \qquad (14)$$

$$\left[\sigma_{Qy}{}^2\right]_{wav}=Q^2\frac{1}{6}\left(\frac{\Delta\lambda}{\lambda}\right)^2.$$

In summary, the Q resolution is then obtained as:

$$\sigma_{Qy}^{2} = \left[\sigma_{Qy}^{2}\right]_{geo} + \left[\sigma_{Qy}^{2}\right]_{wav} + \left[\sigma_{Qy}^{2}\right]_{grav} \quad (15)$$

$$\sigma_{Qx}^{2} = \left(\frac{2\pi}{\lambda L_2}\right)^2 \left[\left(\frac{L_2}{L_1}\right)^2 \frac{R_1^{2}}{4} + \left(\frac{L_1+L_2}{L_1}\right)^2 \frac{2}{3}\left(\frac{\Delta\lambda}{\lambda}\right)^2 \frac{R_2^{2}}{4} + \frac{1}{3}\left(\frac{\Delta x_3}{2}\right)^2\right] + Q_x^{2}\frac{1}{6}\left(\frac{\Delta\lambda}{\lambda}\right)^2$$

$$\sigma_{Qy}^{2} = \left(\frac{2\pi}{\lambda L_2}\right)^2 \left[\left(\frac{L_2}{L_1}\right)^2 \frac{R_1^{2}}{4} + \left(\frac{L_1+L_2}{L_1}\right)^2 \frac{2}{3}\left(\frac{\Delta\lambda}{\lambda}\right)^2 \frac{R_2^{2}}{4} + \frac{1}{3}\left(\frac{\Delta y_3}{2}\right)^2 + A^2\lambda^4 \frac{2}{3}\left(\frac{\Delta\lambda}{\lambda}\right)^2\right] + Q_y^{2}\frac{1}{6}\left(\frac{\Delta\lambda}{\lambda}\right)^2$$

Using focusing lenses modifies the "sample" term only (second term proportional to R_2^2). This term becomes much smaller with lenses. When lenses are used, the sample aperture R_2 can be made larger without much resolution penalty.

3. MINIMUM Q WITH FOCUSSING LENSES

The minimum reachable value of Q starts at the edge of the beam spot. The geometry with focusing lenses is characterized by an umbra only (with no penumbra). The neutron beam spot at the detector is therefore characterized by a box (not a trapezoidal) profile. A simple optics argument gives for the edge of the beam umbra in the horizontal and vertical directions for each wavelength λ the following:

$$X_{min}(\lambda) = \frac{L_2}{L_1} R_1 + \left(\frac{L_1+L_2}{L_1}\right)\left|1-\left(\frac{\lambda}{\lambda_0}\right)^2\right| R_2 + \frac{\Delta x_3}{2} \quad (16)$$

$$Y_{min}(\lambda) = \frac{L_2}{L_1} R_1 + \left(\frac{L_1+L_2}{L_1}\right)\left|1-\left(\frac{\lambda}{\lambda_0}\right)^2\right| R_2 + \frac{\Delta y_3}{2} + 2A\lambda^2\left(\frac{\Delta\lambda}{\lambda}\right).$$

The last term in $Y_{min}(\lambda)$ is due to gravity effect. Now the minimum achievable spot sizes are obtained by considering the part of the spot due to a wavelength spread Δλ.

$$X_{min}(\lambda) = \frac{L_2}{L_1} R_1 + \left(\frac{L_1+L_2}{L_1}\right)\left|\left(1-\left(\frac{\lambda_0+\Delta\lambda}{\lambda_0}\right)^2\right) - \left(1-\left(\frac{\lambda_0}{\lambda_0}\right)^2\right)\right| R_2 + \frac{\Delta x_3}{2}. \quad (17)$$

The magnitude part reduces to:

$$\left|\left(1-\left(\frac{\lambda_0+\Delta\lambda}{\lambda_0}\right)^2\right) - \left(1-\left(\frac{\lambda_0}{\lambda_0}\right)^2\right)\right| \cong 2\left(\frac{\Delta\lambda}{\lambda}\right). \quad (18)$$

Now that the wavelength averaging has been performed, the 0 subscript in λ_0 is dropped for simplicity. The horizontal and vertical beam spot sizes are:

$$X_{min} = \frac{L_2}{L_1} R_1 + \frac{L_1 + L_2}{L_1} 2\left(\frac{\Delta\lambda}{\lambda}\right) R_2 + \frac{\Delta x_3}{2} \tag{19}$$
$$Y_{min} = \frac{L_2}{L_1} R_1 + \frac{L_1 + L_2}{L_1} 2\left(\frac{\Delta\lambda}{\lambda}\right) R_2 + \frac{\Delta y_3}{2} + 2A\lambda^2\left(\frac{\Delta\lambda}{\lambda}\right).$$

The corresponding values of the minimum Q are:

$$Q_{min}{}^{X} = \frac{2\pi}{\lambda}\frac{X_{min}}{L_2} \tag{20}$$
$$Q_{min}{}^{Y} = \frac{2\pi}{\lambda}\frac{Y_{min}}{L_2}$$

4. MEASURED SANS RESOLUTION

Specific Instrument Configuration

Consider the following instrument configuration:

L_1 = 16.14 m
L_2 = 13.19 m
R_1 = 0.715 cm
R_2 = 0.635 cm
$\Delta x_3 = \Delta y_3$ = 0.5 cm

$$\frac{\Delta\lambda}{\lambda} = 0.13. \tag{21}$$

This gives A = 0.01189 cm/ Å^2.

Measurements with Focusing Lenses

Neutron optics measurements were made using a set of 7 consecutive biconcave MgF_2 lenses (described in a previous chapter) inserted in the beam just before the sample aperture. This set corresponds to a focal wavelength λ_0 around 17.36 Å.

The measured position of the neutron beam spot on the detector agrees with predictions.

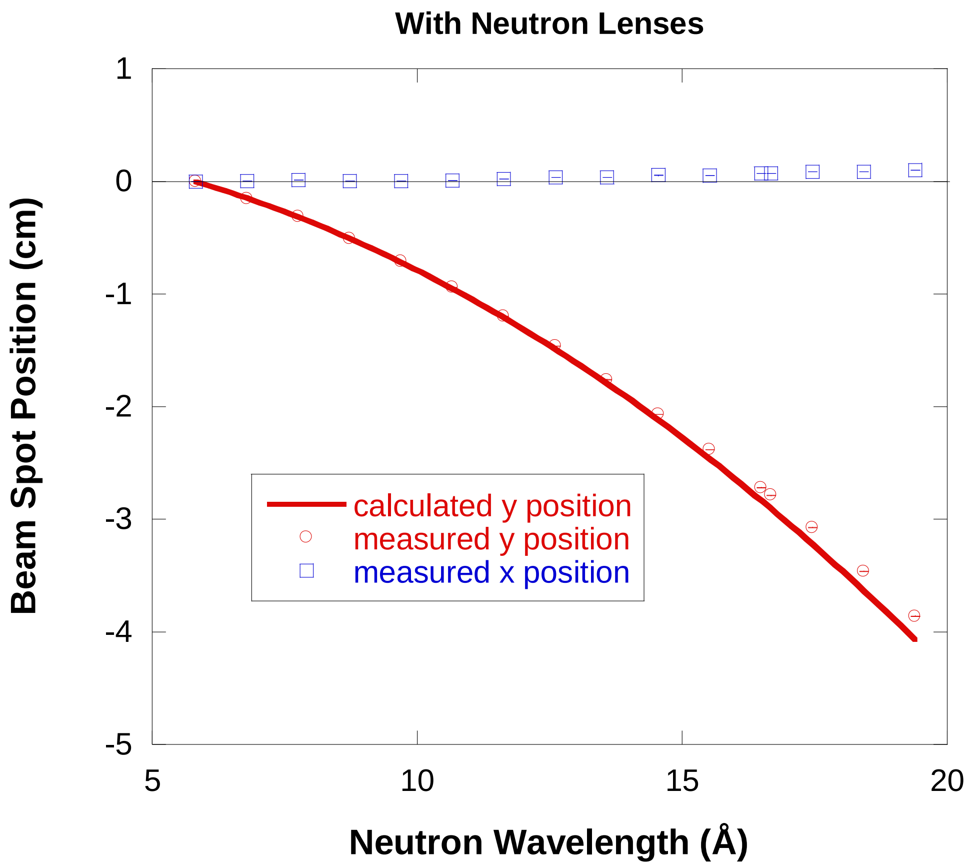

Figure 4: Variation of the neutron beam spot positions with wavelength.

The beam spot resolution has strong (parabolic) wavelength dependence both in the x and in the y directions. The minimum resolution in the horizontal direction corresponds to a focal wavelength λ_0. The minimum in the x direction (λ_0 = 17.2 Å) is taken to be the focal wavelength for our focusing arrangement since the x direction is independent of gravity effects. A procedure of using slice cuts across the beam spot was used to obtain these plotted results (including the $\sqrt{1.45}$ scaling discussed in a previous chapter). The calculated trends agree with the measured ones.

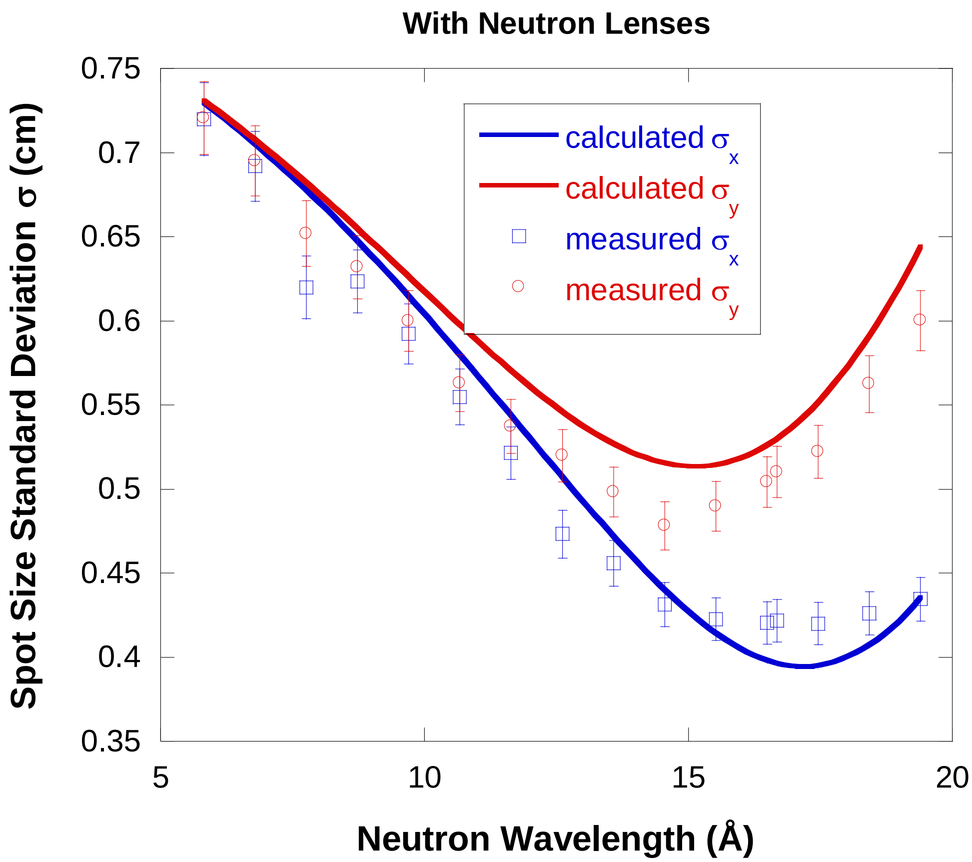

Figure 5: Variation of the spot size standard deviation with wavelength in the horizontal and vertical directions.

Variation of the minimum spot sizes as a function of increasing wavelength is characterized by a minimum around λ_0 = 17.2 Å. The measured values have been chosen conservatively and are found to be overestimates that are higher than the calculated values.

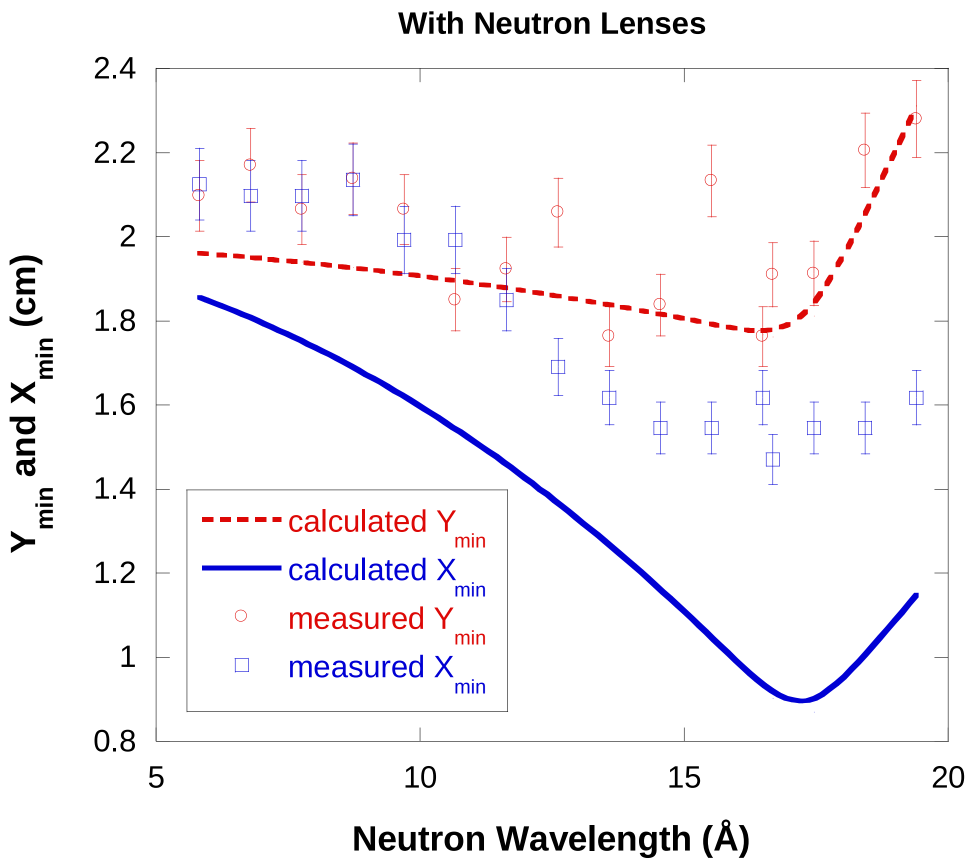

Figure 6: Variation of the minimum spot sizes with increasing wavelength.

Discussion

The use of converging lenses has the advantage of allowing the opening up of the sample aperture (i.e., increasing R_2) without penalty in resolution. This happens because the penumbra is minimized when lenses are used. The main effect is increased neutron current on sample.

Refractive lenses are characterized by chromatic aberrations that show up as a dependence of both the variance σ_x^2 and X_{min} on $(\Delta\lambda/\lambda)$. In order to reduce these chromatic aberrations, $(\Delta\lambda/\lambda)$ could be made smaller; which would result in a penalty in neutron current on sample. Focusing devices that use reflection (rather then refraction) optics (such as elliptical or torroidal mirrors) are not hampered by such chromatic aberrations.

5. LENS TRANSMISSION

The transmission of a set of 7 concave spherical lenses is calculated and compared to transmission measurements. Consider a lens of spherical radius R and thickness 2h at the center and assume that the beam defining aperture has a radius of B.

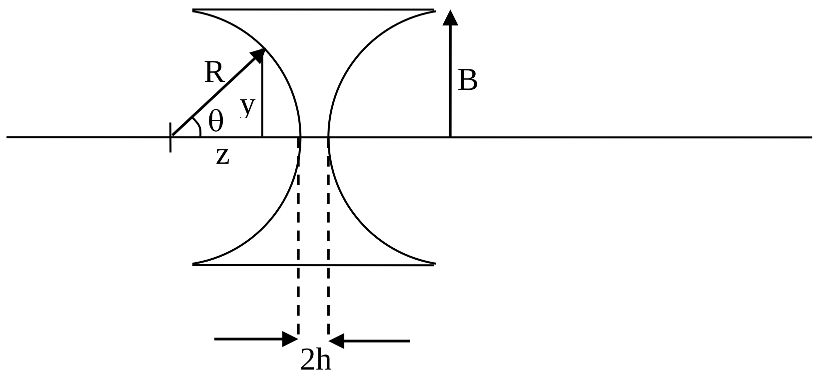

Figure 7: Schematics of the lens geometry.

The transmission of one focusing lens averaged over the beam aperture is given by:

$$T_1 = \frac{1}{\pi B^2} \int_0^B dy 2\pi y \exp[-2\Sigma_t (h + R - z)] \quad (22)$$

Here y is the vertical coordinate, z is the horizontal coordinate obeying $z = \sqrt{R^2 - y^2}$ and Σ_t is the macroscopic cross section for MgF_2. Note that Σ_t varies with neutron wavelength as $\Sigma_t = 0.000513\ \lambda$ where λ is in Å and Σ_t in mm^{-1}. This variation was measured using a uniform thickness slab of MgF_2.

Performing the simple integration, one obtains:

$$T_1 = \frac{\exp[-2\Sigma_t (h + R)]}{2(\Sigma_t B)^2} \left\{ \left[1 - 2\Sigma_t \sqrt{R^2 - B^2}\right] \exp\left[2\Sigma t \sqrt{R^2 - B^2}\right] - (1 - 2\Sigma_t R) \exp[2\Sigma_t R] \right\} \quad (23)$$

The transmission of a set of 7 focusing lenses is given by $T_7 = T_1^7$.

The calculated and measured transmissions for the 7-lens system are compared for increasing neutron wavelength.

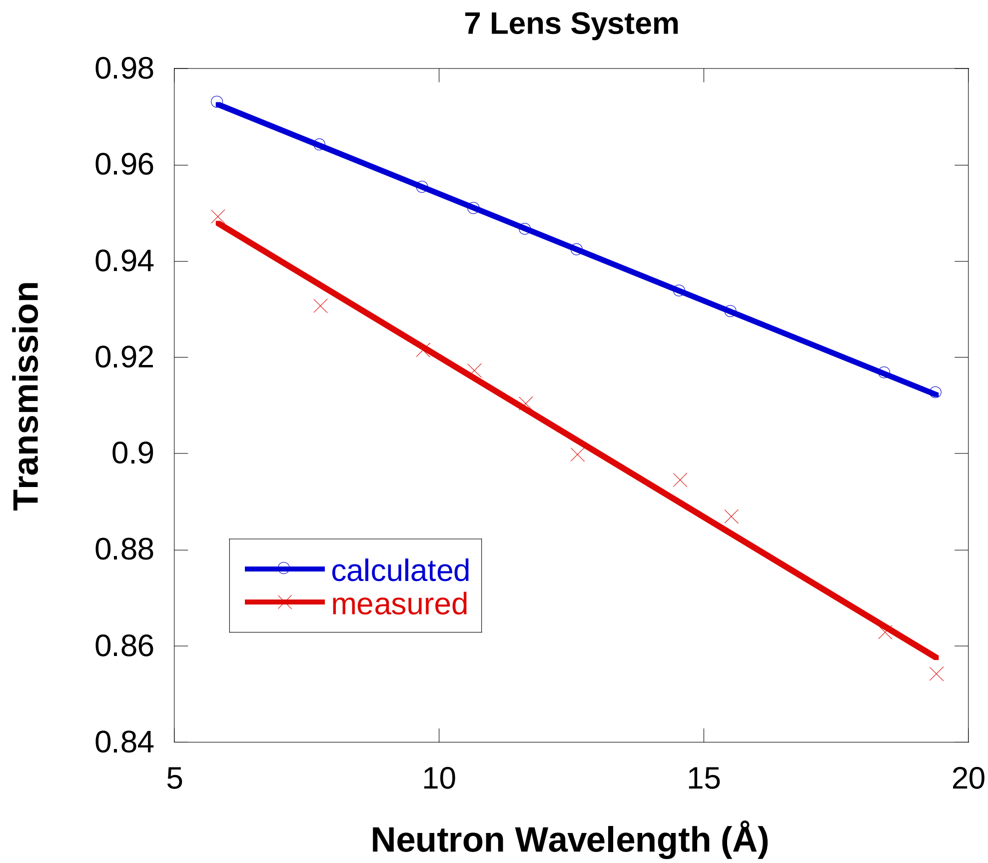

Figure 8: Calculated and measured neutron transmissions for a 7-lens system.

The calculated and measured transmissions agree only partially.

REFERENCES

D.F.R. Mildner, B. Hammouda, and S.R. Kline, "A Refractive Focusing Lens System for SANS", J. Appl. Cryst. 38, 979-987 (2005).

B. Hammouda and D.F.R. Mildner, "SANS Resolution with Refractive Optics", J. Appl. Cryst. 40, 250-259 (2007).

QUESTIONS

1. What is the main difference between focusing lenses for neutrons and focusing lenses for light?
2. Name a typical neutron focusing lens material.
3. When using neutron focusing lenses, what term of the instrumental resolution variance is modified? What is the advantage of this?
4. What are chromatic aberrations?

5. Do reflective optical devices suffer from chromatic aberrations? Name a refractive optics focusing device.
6. Given the transmission T_1 of one focusing lens, calculate the transmission T_7 of a 7-lens system.
7. Using many lens systems, could one build a neutron microscope?
8. What are the two main figures of merit for making a good refractive material to be used for making neutron lenses?

ANSWERS

1. Focusing lenses for neutrons are concave. Focusing lenses for light are convex. This is due to the fact that the index of refraction for neutrons is less than one while that for light is greater than one for most typical focusing materials. This is due to the fact that the scattering length for most materials is positive. Exceptions include hydrogen which has a negative scattering length.
2. MgF_2 is a commonly used neutron focusing lens material.
3. The use of focusing lenses modifies the "sample aperture" term of the resolution variance. This term becomes much smaller even for larger source apertures. The advantage is a larger neutron current on sample.
4. Chromatic aberrations correspond to the de-focusing effect for different wavelengths. The position of the source aperture image changes with wavelength thereby "blurring" the "image".
5. There are no chromatic aberrations with refractive optics. Torroidal or elliptical mirrors are typical refractive optics focusing devices.
6. The transmission of a 7-lens system is given by $T_7 = T_1^7$ where T_1 is the transmission of one lens.
7. If one had lenses after the sample, one could obtain magnification using a neutron beam (neutron microscope). Given the low neutron wavelengths λ (compared to light) the focal length f is very long ($f = \pi R / N\rho b \lambda^2$). Chromatic aberrations, the required long flight paths and coarse detector resolution give only modest magnification and a fuzzy picture. Note that the magnification factor can be worked out to be $M = \frac{f}{L_1 - f}$ where f is the focal length and L_1 is the object (sample in this case)-to-lenses distance. Note that $L_1 = f$ would yield high magnification. However, this condition would require that the lenses-to-image distance L_4 be infinite (recall that $1/L_4 = 1/f - 1/L_1$). This is not realistic.
8. The two figures of merit for refractive materials for making neutron lenses are as follows. (1) High density ρ and high coherent scattering length b in order to make the index of refraction n as small as possible. Recall that $1 - n = \frac{\rho b}{2\pi}\lambda^2$. Making 1-n large (i.e., n small) reduces the focal distance f since $f = R/2N(1-n)$ where R is the lens radius and N is the number of lenses. (2) One would want to minimize the incoherent and absorption scattering cross sections Σ_i and Σ_a in order to minimize background and maximize lens transmission.